Determination of Chiral Perturbation Theory low energy constants from a precise description of pion-pion scattering threshold parameters

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We determine the values of the one and two loop low energy constants appearing in the Chiral Perturbation Theory calculation of pion-pion scattering. For this we use a recent and precise sum rule determination of scattering lengths and slopes that appear in the effective range expansion. In addition we provide new sum rules and the values for these coefficients up to third order in the expansion. Our results when using only the scattering lengths and slopes of the S, P, D and F waves are consistent with previous determinations, but seem to require higher order contributions if they are to accommodate the third order coefficients of the effective range expansion.

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I. INTRODUCTION

The smallness of the u and d quark masses together with the spontaneous $SU(2)$ chiral symmetry breaking of QCD, which implies the existence of the corresponding Goldstone Bosons—the pions—allow us to write a low energy effective theory for QCD, organized as a systematic expansion in pion masses and momenta. This is known as Chiral Perturbation Theory (ChPT) [1, 2], built as the most general low energy expansion of a Lagrangian containing just pions, which is compatible with the symmetry constraints of QCD. In particular, the first order is determined by the scale of the spontaneous symmetry breaking, identified with the pion decay constant f_π, and the pion mass M_π. The expansion is then carried out in powers of $p^2/(4\pi f_\pi^2)$, where p denotes generically either the pion mass or momenta. The details of the QCD underlying dynamics are encoded in a set of low energy constants (LECs), which multiply the independent terms that appear in the Lagrangian at higher orders. Note that all loop divergences appearing in a calculation up to a given order can be reabsorbed by renormalization of the LECs up to that order. In this process the LECs acquire a dependence on the renormalization scale μ, which is canceled with that present in the loops. In this way calculations are rendered finite and scale independent to any given order of the expansion.

Only certain combinations of LECs appear in $\pi\pi$ scattering up to a given order. As we commented above, to leading order $O(p^2)$ there are no LECs. Within the SU(2) formalism, to next to leading order (NLO), or $O(p^4)$, which corresponds to a one-loop calculation, only four LECs, called l_1, l_2, l_3 and l_4 appear in the amplitude, although two of them l_3 and l_4, do so through the quark mass dependence of the pion mass M_π and decay constant f_π. To next to next to leading order (NNLO), or $O(p^6)$, six possible independent terms appear [3] multiplied by six constants, $\bar{b}_i$ with $i = 1...6$, which can be reexpanded in powers of the pion mass in terms of the four one-loop l_k and six new NNLO LECs, denoted by r_i [4]. After renormalization these constants develop a scale dependence becoming $l_i^r(\mu)$ and $r_i^r(\mu)$, whereas the $\bar{b}_i$ remain renormalization scale independent.

Concerning the $O(p^4)$ LECs, we refer the reader to [5] for a recent compilation of lattice QCD and to [6] for some other estimates from quark-model-like calculations. It is also worth noticing that the bulk of their values can be explained by the effect of integrating out heavier resonances and, actually, seems to be saturated by the vector multiplets [7], plus a small contribution from scalars above 1 GeV, and an additional sizable contribution from kaons in the case of the SU(2) formalism [8]. Some estimates from resonance saturation have also been obtained for the $O(p^6)$ parameters [4]. However, since perturbative QCD cannot be applied at very low energies, it is particularly difficult to obtain the values of these LECs from first principles and, with few exceptions, the LECs have been determined best from the comparison with experiment [2, 9–11].

Our aim in this work is to use a very recent dispersive analysis of data in [13], which includes, among others, the latest very precise and reliable results on K_{e4} decays from the NA48/2 collaboration[14], in order to determine the values of the $O(p^4)$ and $O(p^6)$ LECs that appear in the $\pi\pi$ scattering amplitude. Since ChPT is a low energy amplitude, obtained as a truncated series in powers of $(p/4\pi f_\pi)^{2n}$, we will compare with data at threshold. In particular, we will obtain the LECs from fits to the coefficients of the momentum expansion of the amplitude around threshold, usually known as the effective range expansion. The coefficients of this expansion, even up to third order, are also becoming reachable for lattice calculations, although still only limited to the highest isospin channels [12].

Thus, in the next section, after introducing the necessary notation, we present the experimental determination made in [13] of the threshold parameters up to second order of the effective range expansion. The most precise way to obtain these parameters is by means of the Froissart Gribov representation and other sum rules,

which we briefly review. Moreover, in this work, we derive a new set of sum rules to calculate the third order threshold parameters, including their uncertainties, up to the F waves. Some of these third order parameters are of relevance to obtain the values of the LECs, since their leading order contribution is directly proportional to combinations of LECs. Actually, we carefully explain, for each threshold parameter, what is its leading order ChPT, and from what part of the calculation stems from. In Sect. III we first perform a fit of some of these parameters using just the $O(p^4)$ ChPT result, paying particular attention to an estimate of the systematic uncertainties in the parameters, which is of relevance for the role they will play in the determination of different LECs. Still, just by trying to describe a few threshold parameters, we are able to show that the $O(p^4)$ approximation is not enough to describe the data at the present level of precision. In Sect. IV we determine the best $\bar{b}_i$ constants and LECs that appear in the two-loop calculation. We will show that one can obtain only a relatively fair description in terms of $\bar{b}_i$ parameters, although the fact that the $\chi^2/d.o.f.$ of the fits is somewhat larger than one, suggests that at the present level of precision, higher order contributions seem to be required. In Sect. V, we briefly discuss and summarize our results.

II. THRESHOLD PARAMETERS

A. Notation

The amplitude for $\pi\pi$ scattering is customarily decomposed in terms of partial waves t_ℓ^I, of definite isospin I and angular momentum ℓ, as follows:

$$F^I(s,t) = \frac{8}{\pi}\sum_\ell (2\ell+1)\, t_\ell^I(s)\, P_\ell(\cos\theta), \quad (1)$$

$$t_\ell^I(s) = \frac{1}{64\pi}\int_{-1}^{1} T^I(s,t,u) P_\ell(\cos\theta)\, d(\cos\theta), \quad (2)$$

θ being the scattering angle, P_ℓ the Legendre polynomials, s, t, u the usual Mandelstam variables satisfying $s+t+u = 4M_\pi^2$, and T stands for the amplitude. In the elastic regime, the partial waves are uniquely determined by the phase shifts δ_ℓ^I as follows:

$$t_\ell^I(s) = \frac{e^{\delta_\ell^I(s)}\, \sin\delta_\ell^I(s)}{\sigma(s)}, \quad (3)$$

where $\sigma = 2\,p/\sqrt{s} = \sqrt{1-4M_\pi^2/s}$ and p is the CM momentum. With this normalization, the effective range expansion of the real part of a partial wave can be written as:

$$\frac{1}{M_\pi}\mathrm{Re}\, t_\ell^I(s) = p^{2\ell}\Big(a_{\ell I} + b_{\ell I}\, p^2 + \frac{1}{2}c_{\ell I}\, p^4 + ...\Big), \quad (4)$$

where the $a_{\ell I}$ are usually called scattering lengths, the $b_{\ell I}$ slope parameters, the $c_{\ell I}$ shape parameters, and all of them, generically, threshold parameters. Let us remark that it is usual to provide the values of these parameters *in units of* M_π, and so we will do in what follows. In addition, for odd waves we will drop the isospin index, since it can only be $I = 1$ due to Bose symmetry. Finally, we will also make use of the standard spectroscopic notation, where the $\ell = 0, 1, 2, 3...$ are called S,P,D,F... waves, followed by the value of the isospin. As it is customary, when the angular momentum is odd we omit the isospin, since it is fixed to one by Bose symmetry.

	$O(p^2)$	$O(p^4)$		$O(p^6)$		
	pol.	l_i pol.	J	r_i pol.	$l_i\ J$	J^2, K
a_S	x	x	x	x	x	x
b_S	x	x	x	x	x	x
c_S		x	x	x	x	x
a_P	x	x	x	x	x	x
b_P		x	x	x	x	x
c_P			x	x	x	x
a_D		x	x	x	x	x
b_D			x	x	x	x
c_D			x		x	x
a_F			x	x	x	x
b_F			x		x	x
c_F			x		x	x

TABLE I: Contribution to threshold parameters from different orders and kinds of terms within ChPT, as explained in the text. Recall that, due to Bose symmetry, those for S and D waves may have either isospin 0 or 2, whereas those for P and F necessarily have isospin 1.

B. Structure of ChPT calculations

At this point it is relevant to discuss how the different orders of ChPT contribute to each threshold parameter studied here, which we have gathered in the first column of Table I. Let us start with the leading order, $O(p^2)$, of the ChPT amplitude, which is a first order polynomial in terms of Mandelstam variables s, t, u and M_π^2. Since s is independent of θ whereas t and u are first order polynomials in $\cos\theta$, the LO ChPT can only contribute to the a coefficients of the S and P waves and the b coefficients of the S waves, but nothing to any other wave. This corresponds to the second column in Table I.

If we now consider the $O(p^4)$ amplitude, we find two kinds of terms. First, a polynomial, which includes the $l_i(\mu)$ LECs and contains up to two powers of $\cos\theta$, so that it contributes to the a coefficients of S, P and D waves, the b coefficients of the S and P waves as well as the c coefficients of the S waves. This is the third column in Table I. However, there is another kind of $O(p^4)$ contributions, which we have represented in the fourth column. These come from the loop functions, called $J(q^2)$, with two intermediate pions exchanged in any channel. These loop functions carry a non-polynomial dependence on t and u and therefore contribute to all waves, but note that they do not depend on $l_i(\mu)$. This is the fourth column.

labeled "J", in Table I.

Next, to two loops, $O(p^6)$, we find three kinds of terms: First, pure polynomial terms containing the r_i LECs, which can contribute to the a coefficients up to the F wave, b coefficients up to D waves, and c coefficients up to P waves. These appear in the fifth column of Table I. In addition there are terms contributing to all waves, as shown in column six, containing a single one-loop function and $l_i(\mu)$ LECs. Finally, there are also terms without LECs, which correspond to the last column, containing either two one-loop functions or one two-loop function, that we generically call $K(q^2)$.

Therefore, we see that only the a_S, b_S and a_P have a leading order contribution independent of the LECs. A priori, one could expect that the best observables to determine the l_i are those whose leading term is $O(p^4)$, as it is the case of the a_{D0} and a_{D2}, which, actually, have been frequently used to determine the value of l_1 and l_2 [2]. Nevertheless, let us remark that there are still other threshold parameters whose leading contributions are proportional to a combination of l_i, namely, b_P, c_{S0} and c_{S2}. However, these are much harder to determine reliably from experiment and had not been used so far. For this reason, in the next subsection we will provide two new sum rules for their determination. To extract the $O(p^6)$ contributions and the r_i LECs is more difficult, not only because they are generically an smaller effect, but also because they appear together with $O(p^4)$ terms or with $O(p^6)$ terms containing the l_i and a one-loop function.

C. Threshold parameters from sum rules

As commented above, we are going to use the recent and simple data parametrizations obtained in [13]. The relevance of those parametrizations is that they are obtained from data fits which have been highly constrained to satisfy three sets of dispersion relations within uncertainties: Forward dispersion relations up to 1420 MeV, and Roy equations as well as once subtracted Roy-like equations up to 1100 MeV. Above 1420 MeV, Regge expressions, assuming factorization, where fitted to NN, $N\pi$ and $\pi\pi$ total cross sections, and used inside the integrals, allowing for the variation of the Regge parameters within the constrained fits to data. In that work, the values of the a and b threshold parameters were also provided for the S0, S2, P, D0, D2 and F waves, namely, all the combinations of $I = 0, 1, 2$ and $\ell = 0, 1, 2, 3$ allowed by Bose symmetry when considering pions as identical particles. With the aim of minimizing the uncertainties, they were obtained from sum rules, with the only exception of the $5a_{S0} + 2a_{S2}$ combination, which is orthogonal to the one appearing in the Olsson sum rule. The results from sum rules were consistent with, but more accurate than, those directly obtained from the simple phenomenological parametrizations. Let us nevertheless remark that the sum rules used in [13] as well as those we will describe below have a very small dependence on the high energy region.

These results provided us with 12 observables determined from experiment, which we list in Table II, that we want to fit using four l_i LECs in the one-loop case and six parameters $\bar{b}_i$ in the two-loop case, which can be parametrized in terms of ten LECs. Moreover, in order to enlarge the set of observables that we include in our fit, we will also provide here the calculation of the third order coefficient c of the effective range expansion in Eq.(4) above. These are five additional observables.

For this purpose we will use the Froissart-Gribov sum rules, which, for $\ell > 0$ allow us to write the c parameters as:

$$c_{\ell I}=\frac{\sqrt{\pi}\,\Gamma(\ell+1)}{M_\pi\,\Gamma(\ell+3/2)}\int_{4M_\pi^2}^{\infty}ds\left\{\frac{16\,\mathrm{Im}\,F^{I''}(s,4M_\pi^2)}{(s-4M_\pi^2)^2 s^{\ell+1}}-8(\ell+1)\frac{\mathrm{Im}\,F^{I'}(s,4M_\pi^2)}{(s-4M_\pi^2)s^{\ell+2}}+\frac{\mathrm{Im}\,F^{I}(s,4M_\pi^2)}{s^{\ell+3}}\frac{(\ell+2)^2(\ell+1)}{\ell+3/2}\right\}, \quad (5)$$

where the primes denote the derivative with respect to $\cos\theta$. These formulas allow us to calculate the c parameters for the P, D0, D2 and F wave, that we list in Table II. Note that the resulting values are all very accurate with the exception of the c_P coefficient, and that the above sum rules are not applicable to the scalar case. These are the reasons why we provide here three new sum rules, one for c_P and two for c_{S0} and c_{S2},

$$c_P=-\frac{14\,a_F}{3}+\frac{16}{3M_\pi}\int_{4M_\pi^2}^{\infty}ds\left\{\frac{\mathrm{Im}\,F^{I=0}(s)}{3s^4}-\frac{\mathrm{Im}\,F^{I=1}(s)}{2s^4}-\frac{5\mathrm{Im}\,F^{I=2}(s)}{6s^4}+\left[\frac{\mathrm{Im}\,F^{I=1}(s)}{(s-4M_\pi^2)^4}-\frac{3a_P^2M_\pi}{4\pi(s-4M_\pi^2)^{3/2}}\right]\right\}, \quad (6)$$

$$c_{S2} = -6b_P - 10a_{D2} + \frac{8}{M_\pi}\int_{4M_\pi^2}^\infty ds \left\{ \frac{\mathrm{Im}\, F^{0+}(s)}{s^3} + \frac{1}{(s-4M_\pi^2)^{5/2}} \times \left[\frac{\mathrm{Im}\, F^{0+}(s)}{\sqrt{s-4M_\pi^2}} - \frac{2M_\pi a_{S2}^2}{\pi} - \frac{s-4M_\pi^2}{\pi}\left(\frac{M_\pi}{2}(2a_{S2}b_{S2}+a_{S2}^4) - \frac{a_{S2}^2}{4M_\pi}\right)\right]\right\}, \tag{7}$$

$$c_{S0} = -2c_{S2} - 20a_{D2} - 10a_{D0} + \frac{12}{M_\pi}\int_{4M_\pi^2}^\infty ds \left\{ \frac{\mathrm{Im}\, F^{00}(s)}{s^3} + \frac{1}{(s-4M_\pi^2)^{5/2}} \times \left[\frac{\mathrm{Im}\, F^{00}(s)}{\sqrt{s-4M_\pi^2}} - \frac{4M_\pi(2a_{S2}^2+a_{S0}^2)}{3\pi} - \frac{s-4M_\pi^2}{3\pi}\left(M_\pi[2(2a_{S2}b_{S2}+a_{S2}^4) + 2a_{S0}b_{S0}+a_{S0}^4] - \frac{2a_{S2}^2+a_{S0}^2}{2M_\pi}\right)\right]\right\} \tag{8}$$

Let us note that, in all these sum rules, we have written several terms together inside square brackets to emphasize that they do not converge separately. The derivation is similar to the sum rules obtained for b_P, b_{S0} and b_{S2} in [15, 16]. They correspond to the threshold limit, taken from above, of the second derivative of a forward dispersion relation for the $F^{I=1}$, F^{0+} and F^{00} amplitudes, respectively. Let us recall that $F^{0+} = F^{I=2}/2 + F^{I=1}/2$ whereas $F^{00} = 2F^{I=2}/3 + F^{I=0}/3$.

In the above sum rules we have explicitly got rid of the principal part that appears in the dispersion relation by using:

$$\mathrm{P.P.}\int_0^\infty \frac{dx}{(x-y)\sqrt{x}} = 0, \ \mathrm{for}\, y>0. \tag{9}$$

III. $O(p^4)$ FITS

Before presenting the fits to the full two-loop ChPT results, it is instructive to try to fit the threshold parameters by means of the one-loop ChPT amplitudes. This will help us check the stability of the LECs values and the need for higher order counterterms, but it will also help us illustrate our different fitting strategies in order to deal with systematic uncertainties.

Let us recall once more that to $O(p^4)$ only four LECs, appear in $\pi\pi$ scattering, customarily denoted by $\bar{l}_1, ..., \bar{l}_4$, which are basically the $l_i^r(\mu)$ at the $\mu = M_\pi$ scale and normalized so that they have values of order one [2]. Note, however, that $\bar{l}_3$ and $\bar{l}_4$ only appear through the quark mass dependence of M_π and f_π, respectively, and therefore we cannot expect much sensitivity to these two parameters from fits to the coefficients of the momentum expansion of amplitudes.

In addition, from Table I, we see that, up to $O(p^4)$, only ten observables carry any dependence on the LECs: for half of them, a_{S0}, a_{S2}, a_P, b_{S0} and b_{S2}, the leading contribution is $O(p^2)$, whereas for the other five the leading contribution is directly of $O(p^4)$. Therefore, we expect the latter to be more sensitive to the LECs, but also to the higher order corrections that we are neglecting.

	CFD	Sum rules	Best value
a_{S0}	0.221 ± 0.009		0.220 ± 0.008
$a_{S2}(\times 10^2)$	-4.3 ± 0.8		-4.2 ± 0.4
$2a_{S0} - 5a_{S2}$	0.657 ± 0.043	0.648 ± 0.016	0.650 ± 0.015
$a_P(\times 10^3)$	38.5 ± 1.2	37.7 ± 1.3	38.1 ± 0.9
$a_{D0}(\times 10^4)$	18.8 ± 0.4	17.8 ± 0.3	17.8 ± 0.3
$a_{D2}(\times 10^4)$	2.8 ± 1.0	1.85 ± 0.18	1.85 ± 0.18
$a_F(\times 10^5)$	5.1 ± 1.3	5.65 ± 0.23	5.65 ± 0.23
b_{S0}	0.278 ± 0.007	0.278 ± 0.008	0.278 ± 0.005
$b_{S2}(\times 10^2)$	-8.0 ± 0.9	-8.2 ± 0.4	-8.2 ± 0.4
$b_P(\times 10^3)$	5.07 ± 0.26	6.0 ± 0.9,5.48 ± 0.17	5.37 ± 0.14
$b_{D0}(\times 10^4)$	-4.2 ± 0.3	-3.5 ± 0.2	-3.5 ± 0.2
$b_{D2}(\times 10^4)$	-2.8 ± 0.8	-3.3 ± 0.1	-3.3 ± 0.1
$b_F(\times 10^5)$	-4.6 ± 2.5	-4.06 ± 0.27	-4.06 ± 0.27
$c_{S0}(\times 10^2)$	-0.12 ± 1.22	0.7 ± 0.8	0.45 ± 0.67
$c_{S2}(\times 10^2)$	3.6 ± 1.8	2.79 ± 0.24	2.80 ± 0.24
$c_P(\times 10^3)$	1.41 ± 0.19	2.3 ± 0.8,1.35 ± 0.15	1.39 ± 0.12
$c_{D0}(\times 10^4)$	5.6 ± 0.4	4.4 ± 0.3	4.4 ± 0.3
$c_{D2}(\times 10^4)$	5.5 ± 1.6	3.6 ± 0.2	3.6 ± 0.2
$c_F(\times 10^5)$	11 ± 9	6.9 ± 0.4	6.9 ± 0.4

TABLE II: Values of threshold parameters obtained in Ref. [13] together with those obtained here for the c parameters. The "CFD" column lists the values as obtained directly from the Constrained Data Fits provided in [13]. We also provide the values obtained from sum rules. We typically consider these our best results, except in cases when the CFD are competitive and not very correlated with the sum rule. Note that for b_P and c_P waves we provide two values. For c_P, the first one, less accurate, corresponds to the Froissart-Gribov sum rule in Eq.(5) and the second one to the sum rule in Eq.(6). Similarly, for b_P, the first, less precise result, is from the Froissart-Gribov sum rule, and the second from a fast convergent sum rule, as explained in [15, 16].

Thus, in Table III we show the results of our fits. First, we have fitted only the observables which have a leading $O(p^2)$ contribution, since, in principle these might be more stable under the higher order corrections. The fit comes out with relatively low $\chi^2/d.o.f.$. Next we have presented a determination of $\bar{l}_1$ and $\bar{l}_2$, which are, in prin-

ciple, fixed from a_{D0} and a_{D2} alone, which are not included in the previous fit. It is evident that the resulting values from those two fits are incompatible, particularly $\bar{l}_1$. The incompatibility is even worse when fitting simultaneously the ten observables that depend on $\bar{l}_i$ to $O(p^4)$. These results imply that, as is well known, to the present level of precision the one-loop ChPT formalism is not enough and calls for higher order corrections.

For instance, the effect of higher order corrections can be seen by fitting to the one-loop amplitude but replacing f_π by f_0 in the $O(p^4)$ terms, since the resulting expression is also correct up to $O(p^4)$, only differing in higher order contributions. This we show in row 5 of Table III. Surprisingly the $\chi^2/d.o.f$ comes somewhat lower, but the values of the LECs come out rather different from the previous calculation.

Of course, one could always try to include the effect of higher orders into a systematic uncertainty of the LECs, if one still wants to use the relatively simple $O(p^4)$ approximation instead of the full two-loop amplitude, at the expense of accuracy. In such case we propose to take the weighted average of the two previous fits, including a systematic uncertainty to cover the LECs values of both fits [1]. In Table IV we compare the resulting threshold parameters obtained using this averaged set with the Best results of Table II, which we repeat under the "Data analysis" column. We can see there that, thanks to the larger uncertainty, the threshold parameters obtained are compatible within errors with the experimental values, except for b_{S0} and b_P, which differ by more than three and two standard deviations respectively.

IV. $O(p^6)$ FITS

As already commented in the introduction, the threshold parameters can be described in ChPT at $O(p^6)$ in terms of six low energy constants, usually denoted $\bar{b}_1, \cdots, \bar{b}_6$. Let us remark, however, that the first four can be separated in two parts with different chiral order, namely, $\bar{b}_i = \bar{b}_i^{(0)} + \Delta\bar{b}_i$, $i = 1,2,3,4$, where $\bar{b}_i^{(0)} = O(m_\pi^0)$ and $\Delta\bar{b}_i = O(m_\pi^2)$. The $\bar{b}_i^{(0)}$ parameters contain combinations of the four $O(p^4)$ LECs $\bar{l}_i$, but not of the $O(p^6)$ LECs. In contrast, six linear combinations of the latter appear inside the $\Delta\bar{b}_i$ for $i = 1...4$ as well as in $\bar{b}_5$ and $\bar{b}_6$, and are accordingly denoted by r_i, with $i = 1...6$. Due to this $O(m_\pi^2)$ part in the parameters, the calculations using the $\bar{b}_i$ have an extra $O(p^8)$ piece which is not present when using $\bar{l}_i$ and r_i (or making the separation $\bar{b}_i = \bar{b}_i^{(0)} + \Delta\bar{b}_i$ explicit). Of course, since this is a higher order contribution, both descriptions are formally equivalent up to $O(p^6)$. Nevertheless, there could be relevant numerical differences and, what is more important to us, in one case one should determine only 6 parameters, whereas in the other case there are 10 parameters

Thus, when using $\bar{l}_i$ and r_i, we may obtain spurious solutions or, in general, less stable values than when using just the six $\bar{b}_i$. That is why in this section we have decided to use the $\bar{b}_i$ set in our fits. For completeness we provide in the appendix a detailed account of our results when parametrizing the ChPT series in terms of $\bar{l}_i$ and r_i.

Therefore, and similarly to the $O(p^4)$ case, we first fit the ten threshold parameters a_S, a_P, a_D, b_S, b_P, and c_S, which have a non–zero $O(p^4)$ polynomial contribution, since we expect these to be more stable under higher order ChPT corrections. In the first row of Table V we show the resulting $\bar{b}_i$ for this fit, which describes fairly well the fitted observables with a $\chi^2/d.o.f. = 1.2$. However, the threshold parameters which are not fitted, are not so well described with this set of LECs.

Actually, when fitting all 18 threshold parameters, we obtain somewhat different LECs, which are shown in the second row of Table V. Although not dramatically incompatible with those of the first row, we see differences around two standard deviations for $\bar{b}_3$ and $\bar{b}_4$ and around three standard deviations for $\bar{b}_1$ and $\bar{b}_5$. Unfortunately, this second fit comes out with a rather poor $\chi^2/d.o.f. = 5.2$. Therefore, it seems that we cannot describe all observables simultaneously with two-loop ChPT within the present level of precision. Higher order contributions seem to be required.

Nevertheless, we have noticed that c_P alone contributes almost to one third of the total χ^2. This might indicate that c_P receives important higher order contributions that are not being taken into account in the $O(p^6)$ calculation. Once again we can obtain a crude estimate of the size of higher order ChPT corrections, by changing f_π by f_0 in the last term of the ChPT expansion. By far it is c_P the one that suffers the largest change, by almost 80%. Certainly it looks as a good candidate to receive very large higher order ChPT corrections.

Thus, we proceed to fit again all threshold parameters except c_P, and the result is shown in the third row of Table V. The fit quality improves sizably, but we still get a high $\chi^2/d.o.f. = 2.9$, which indicates that the two-loop calculation may not be enough to describe even the remaining threshold parameters with their current level of precision.

As a final attempt, we can see the effect of higher order corrections by making a fit replacing f_π by f_0 in the $O(p^6)$ terms, since the resulting expression is also correct up to $O(p^6)$. We show the results (without including c_P) in the fourth row of Table V. Surprisingly, we now obtain a good $\chi^2/d.o.f. = 1.0$, and all LECs are less than two standard deviations from those obtained by fitting only the threshold parameters which have an $O(p^4)$ polynomial part. We therefore conclude that, by excluding c_P, the two-loop fit still shows some tension but by conve-

[1] We have weighted each fit by the square of its $\chi^2/d.o.f.$, whereas the uncertainty is obtained by adding in quadrature the averaged statistical uncertainty to the maximum difference between the resulting central value and the central value of each fit.

Fit to	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_3$	$\bar{l}_4$	$\chi^2/d.o.f.$
a_S, b_S, a_P	1.1±1.0	5.1±0.7	−1±8	7.1±0.7	0.23
a_D	−1.75±0.22	5.91±0.10	—	—	0
c_S	−2.4±0.9	4.8±0.4	—	—	0
$a_S, b_S, a_P, a_D, c_S, b_P$	−2.06±0.14	5.97±0.07	−5±8	7.1±0.6	7.9
$a_S, b_S, a_P, a_D, c_S, b_P$, using f_0	−1.06±0.11	4.6±0.9	0±6	5.0±0.3	7.06
Estimate $O(p^4)$	−1.5±0.5	5.3±0.7	−3±7	6.0±1.2	—

TABLE III: $O(p^4)$ fits to different sets of threshold parameters containing polynomial $O(p^4)$ contributions. Note that the results of the three first lines are rather incompatible with each other. This is also illustrated by the large $\chi^2/d.o.f.$ when fitting all the observables simultaneously. We also show two versions of such a fit, either using f_π or f_0 in the last order or the expansion. Finally, we provide an estimate of how much one should enlarge the uncertainties of the LECs if, for simplicity, one still insists in using the one-loop formalism. Beyond that accuracy a two loop formalism is called for.

niently using f_0 the last term of the ChPT expansion, it can give an acceptable description of the rest of threshold parameters.

For this reason, we have once more made a weighted average of the two fits (the one using f_π and the one using f_0) adding systematic uncertainties to cover both sets. This we show in the fifth row of Table V, where we can see that they are also quite compatible with previous determinations in the literature [11]. Let us remark that, as emphasized in [11], "the error bars only indicate the noise" seen in their evaluation. This would correspond to our uncertainties in the "All but c_P, using f_0" row, whereas the error bars we provide for our final result also contain some crude estimate of higher order uncertainties.

Finally, the values obtained for the threshold parameters using this averaged set of LECs are shown in the second column of Table IV, where we can see that, with the exception of c_P, they are rather compatible with the experimental determination.

	Estimate $O(p^4)$	Estimate $O(p^6)$	Data Analysis
a_{S0}	0.214±0.009	0.230 ±0.014	0.220 ± 0.008
$a_{S2}(\times 10^2)$	-4.4±0.3	-4.3±0.4	-4.2 ± 0.4
$a_P(\times 10^3)$	38.7±1.2	39.0±0.8	38.1 ± 0.9
$a_{D0}(\times 10^4)$	15±3	16.9±0.9	17.8 ± 0.3
$a_{D2}(\times 10^4)$	1.3±1.0	1.7±0.3	1.85 ± 0.18
$a_F(\times 10^5)$	—	4.6±0.5	5.65 ± 0.23
b_{S0}	0.255±0.011	0.271±0.007	0.278 ± 0.005
$b_{S2}(\times 10^2)$	-8.2±0.5	-8.4±0.2	-8.2 ± 0.4
$b_P(\times 10^3)$	4.4±0.5	5.2±0.2	5.37 ± 0.14
$b_{D0}(\times 10^4)$	—	-3.6±0.8	-3.5 ± 0.2
$b_{D2}(\times 10^4)$	—	-3.1±0.4	-3.3 ± 0.1
$b_F(\times 10^5)$	—	-3.4±0.3	-4.06 ± 0.27
$c_{S0}(\times 10^2)$	2.3±1.4	1.3±0.6	0.45 ± 0.67
$c_{S2}(\times 10^2)$	3.4±0.7	2.78±0.16	2.80 ± 0.24
$c_P(\times 10^3)$	—	0.3±0.2	1.39 ± 0.12
$c_{D0}(\times 10^4)$	—	3.6±0.2	4.4 ± 0.3
$c_{D2}(\times 10^4)$	—	3.2±0.2	3.6 ± 0.2
$c_F(\times 10^5)$	—	5.4±0.4	6.9 ± 0.4

TABLE IV: Values of the threshold parameters obtained from $O(p^4)$ ChPT and $O(p^6)$ ChPT using the averaged sets of LECs from the sixth row of Table III and the fifth row of Table V. We also show in the third column the best values obtained from the data analysis given in Table II.

V. SUMMARY AND DISCUSSION

In this work, we have determined the low energy constants of $SU(2)$ Chiral Perturbation Theory (ChPT) at one and two loops from the determinations of the threshold parameters obtained from sum rules using a recent and precise dispersive analysis of data [13], together six additional observables that we have studied here.

Threshold parameters are defined as the coefficients of the effective range expansion of $\pi\pi$ scattering partial waves, which in this work we have studied up to angular momentum $\ell = 3$, i.e., S,P, D and F waves, and all possible isospin states $I = 0, 1, 2$. The coefficients of the two first orders, namely the scattering lengths $a_{\ell I}$ and slope parameters $b_{\ell I}$, were already obtained from a dispersive analysis of data in [13]. In addition, we have provided here three new sum rules to estimate the third order coefficients $c_{\ell I}$, thus adding six new observables to form a total set of eighteen. We have briefly reviewed how the different terms and low energy constants of ChPT contribute to each one of these threshold parameters.

We have then proceeded to fit these observables, first within one-loop ChPT, $O(p^4)$, and then to the full two-loop $O(p^6)$ calculation [4]. We have checked that the one-loop formalism is clearly insufficient to accommodate the present level of precision. There is a clear improvement, in terms of $\chi^2/d.o.f.$ when using the two-loop expansion, although it is still not sufficient to get a good quality fit. This suggests that even higher order ChPT contributions may still be required to describe all these observables

Fit to	$\bar{b}_1$	$\bar{b}_2$	$\bar{b}_3$	$\bar{b}_4$	$\bar{b}_5$	$\bar{b}_6$	$\chi^2/d.o.f.$
$a_S, b_S, a_P, a_D, c_S, b_P$	-14±4	14.6±1.2	-0.29±0.05	0.76±0.02	0.1±1.1	2.2±0.2	6.0/(10-6+1)=1.2
All	-2±3	14.2±1.0	-0.39±0.04	0.746±0.013	3.1±0.3	2.58±0.12	67/(18-6+1)=5.2
All but c_P	-6±3	15.9±1.0	-0.36±0.04	0.753±0.013	2.2±0.4	2.44±0.12	34.9/(17-6+1)=2.9
All but c_P, using f_0	-12±3	13.9±0.9	-0.30±0.04	0.726±0.013	1.0±0.3	1.93±0.08	12.5/(17-6+1)=1.04
Estimate $O(p^6)$	-10.5±5.1	14.5±1.8	-0.31±0.06	0.73±0.02	1.3±1.0	2.1±0.4	—
Ref. [11]	-12.4±1.6	11.8±0.6	-0.33±0.07	0.74±0.01	3.6±0.4	2.35±0.02	

TABLE V: $O(p^6)$ fits to different sets of threshold parameters. In the first row we only fit to observables containing polynomial $O(p^4)$ contributions. Note the improvement of the $O(p^6)$ description versus the $O(p^4)$ one by comparing the $\chi^2/d.o.f.$ here with the corresponding one in Table III. Next we show the results of the fit to all the threshold parameters obtained in this work. Note that the fit quality is rather poor. However, most of the disagreement is caused by a single observable c_P. When this observable is omitted, the resulting fits are of much better quality, particularly good when using f_0 instead of f_π in the last term of the ChPT expansion. We also provide an estimate of the LECs uncertainties from the fits to all observables except c_P, as a weighted average of the fits using f_0 or f_π. Within our uncertainties, the resulting values of the $\bar{b}_i$ parameters are very consistent with previous determinations, listed in the last row. Let us remark that, as emphasized in [11], "the error bars only indicate the noise" seen in their evaluation. This would correspond to our uncertainties in the "All but c_P, using f_0" row, whereas the error bars we provide for our final result also contain some crude estimate of higher order uncertainties.

simultaneously.

However, we have been able to identify that the largest incompatibility is due to the c_P parameter. This may not come as a big surprise, since the largest contribution to the value of the sum rule that determines c_P is given by the ρ resonance and its sharp rise before 770 MeV, which cannot be reproduced by the perturbative ChPT series. Actually, we have estimated, by changing f_π from its physical value to its value in the chiral limit in the last term of the $O(p^6)$ expansion, that this observable is a natural candidate to receive very large corrections form higher ChPT orders.

Hence, if c_P is omitted, the quality of the two loop fit improves, although there is still some tension in the parameters to describe the remaining threshold parameters. Nevertheless, by conveniently using f_0 in the last term of the ChPT expansion, the two -loop expansion can provide an acceptable description of the rest of threshold observables. The ChPT parameters thus obtained, for which we provide statistical as well as an estimate of systematic uncertainties, are fairly compatible with previous determinations.

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Appendix A: Fits to $\bar{l}_i$ and $\tilde{r}_i$

As commented in Section IV, the two-loop $\pi\pi$ scattering amplitudes can be recast in terms of six independent terms multiplied by their corresponding low energy constants $\bar{b}_i$. In turn, these $\bar{b}_i$ can be rewritten in terms of the four $O(p^4)$ LECs that appear in the Lagrangian and six combinations r_i of the $O(p^6)$ LECs. The difference between writing the amplitude in one way or the other is $O(p^8)$. However, despite increasing the number of parameters to ten, the $O(p^6)$ amplitude still provides just six independent structures. As a consequence, the fits in terms of $\bar{l}_i$ and r_i are much more unstable, and can even lead to spurious solutions. For this reason we have explained the fits in terms of $\bar{b}_i$ in the main text, and we have relegated the $\bar{l}_i$, r_i fits to this appendix.

Let us then revisit the fits of Table V where we fit all the observables, or all but c_P, but recasting the amplitudes in terms of $\bar{l}_i$ and $\tilde{r}_i$.[2] The resulting values are given in Table VI. We observe the same pattern as before: the fit to all parameters still has a rather high $\chi^2/d.o.f. = 5$, because, although the total χ^2 has decreased from 67 to 42, the number of degrees of freedom has increased by 4. Let us remark that $\tilde{r}_1$ and $\tilde{r}_2$ have central values many orders of magnitude bigger than expected, but their uncertainties are comparably large. This means that we do not have any real sensitivity to these parameters.

Once again, the largest contribution to the χ^2 is due to c_P, and thus we remove it from the fits, as we did in the main text. When so doing, the $\chi^2/d.o.f.$ becomes less than one, yielding a statistically acceptable fit, but of course, the uncertainties are still huge for $\tilde{r}_1$ and $\tilde{r}_2$ and very large for $\bar{l}_3$, $\tilde{r}_3$ and $\tilde{r}_4$. The central value of $\bar{l}_3$

[2] We give the fit results in terms of $\tilde{r}_i$, defined as $\tilde{r}_i = (4\pi)^4 r_i$, which have a size of order one.

Fit to:	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_3$	$\bar{l}_4$	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$	$\tilde{r}_5$	$\tilde{r}_6$	$\chi^2/d.o.f.$
All	-0.88±1.43	5.1± 0.8	-49±10	4.5±1.3	-984±335	-101±302	-5.7±26	-13±15	1.6±0.9	0.45±0.33	$\frac{42}{18-10+1} = 5$
All but c_P	-2.2±1.5	5.6±0.8	-20±11	10±2	276±845	-1361±549	34±36	-38±19	0.67±0.94	0.66±0.34	$\frac{6.7}{17-10+1} = 0.8$
All but c_P with f_0	-0.5±1.0	4.2±0.6	-6±8	6.6±1.1	46±450	-356±238	4±15	-9±9	1.5±0.6	0.5±0.2	$\frac{4.7}{17-10+1} = 0.6$
Constrained fit to:											
All but c_P	-0.11±0.16	4.2±0.1	3.3	5.8±0.4	-1.5	3.2	-4.2	-2.5	3.1±0.5	0.85±0.15	$\frac{68}{17-5+1} = 5.2$
All but c_P with f_0	0.5±0.2	3.9±0.1	3.3	5.1±0.3	-1.5	3.2	-4.2	-2.5	1.4±0.4	0.47±0.12	$\frac{15.7}{17-5+1} = 1.2$
Estimate $O(p^6)$	0.4±0.5	3.9±0.3	3.3	5.2±0.7	-1.5	3.2	-4.2	-2.5	1.7±1.5	0.5±0.3	—
Estimate $O(p^4)$	-1.5±0.5	5.3±0.7	-3±7	6.0±1.2	—	—	—	—	—	—	—
Refs.[4, 5, 11]	-0.4±0.6	4.3±0.1	3.3±0.7	4.4±0.2	-1.5	3.2	-4.2	-2.5	3.8±1.1	1.0±0.1	—

TABLE VI: $O(p^6)$ fits to different sets of threshold parameters using the low energy constants $\bar{l}_i$ and $\tilde{r}_i$. In the first row we fit all the threshold parameters obtained in this work. Note that the $\chi^2/d.o.f.$ is quite large. However, as it happened in the equivalent fit in Sect. IV, the largest contribution to χ^2 comes from c_P. Thus, in the following fits we include all the observables but c_P. In the second and third rows we show the LECs obtained when c_P is excluded, using f_π and f_0 in the last term of the ChPT expansion, respectively. The quality of the fits notably improves. Nevertheless, the large size of the errors in the case of $\bar{l}_3$, $\tilde{r}_1$, $\tilde{r}_2$, $\tilde{r}_3$ and $\tilde{r}_4$ indicates that our fits are not very sensitive to these LECs. For that reason, in the next section of the table ("Constrained fit to") we repeat the latter two fits, fixing the value of $\bar{l}_3$ to an average of lattice determinations [5] and that of the LECs $\tilde{r}_1$ to $\tilde{r}_4$ to the resonance saturation estimates [4]. We provide an estimate of the LECs and their uncertainties as a weighted average of these two last fits. Let us note that the $O(p^4)$ LECs $\bar{l}_1$ to $\bar{l}_4$ do not lie too far from our $O(p^4)$ estimates, shown in the seventh row. Moreover, within our uncertainties, the resulting values of the $\bar{b}_i$ parameters are very consistent with previous determinations, listed in the last row ($\bar{l}_3$ from [5], $\tilde{r}_1$ to $\tilde{r}_4$ from [4] and the rest of the LECs from [11]).

is also far from our $O(p^4)$ values, also given in Table VI, or the lattice value in [5]. Similarly, those of $\tilde{r}_1$ to $\tilde{r}_4$ are far from the resonance saturation estimates in [4]. However, all of them are still relatively compatible due to the resulting large uncertainties.

As we did in the main text, we repeat this last fit to all parameters but c_P, this time replacing f_π by f_0 in the $O(p^6)$ terms, which is a change of higher order in the ChPT expansion. We observe that we obtain a similarly good description of the observables, but with LECs closer to the reference values from [4, 5, 11].

In fact, since we observe that our fits are not very sensitive to the value of some LECs, we can ask what quality we can achieve if we fix these parameters to the reference values. We thus repeat the fit to all the observables but c_P, fixing $\bar{l}_3$ and $\tilde{r}_1$ to $\tilde{r}_4$ to the reference values given in the last row of Table VI, both using f_π and f_0 in the $O(p^6)$ terms. The results of these "constrained" fits are shown in the fourth and fifth rows of the table. As expected, the $\chi^2/d.o.f$ does not increase much with respect to the unconstrained fit, confirming that our observables depend little on these LECs and thus we are not able to provide an estimate for them. For the rest of LECs, we give as estimates the weighted average of the values found in the constrained fits, all of them fairly compatible within uncertainties with the reference values.

Appendix B: Threshold parameters in ChPT

We show here the two-loop ChPT expressions for the $F-$wave threshold parameters as well as the third order threshold parameters, $c_{\ell I}$, for all waves. The scattering lengths, $a_{\ell I}$, and slope parameters, $b_{\ell I}$, for the S, P and D waves can be found, for instance, in [4].

$$
\begin{aligned}
a_F &= \frac{11}{94080\pi^3 f_\pi^4 M_\pi^3}\left[1+\frac{M_\pi^2}{132\pi^2 f_\pi^2}\left(9\bar{b}_1+\frac{51}{2}\bar{b}_2-151\bar{b}_3-653\bar{b}_4+126\bar{b}_5+126\bar{b}_6+\frac{111\pi^2}{20}+\frac{4111}{9}\right)\right],\\
b_F &= -\frac{47}{529200\pi^3 f_\pi^4 M_\pi^5}\left[1+\frac{M_\pi^2}{752\pi^2 f_\pi^2}\left(75\bar{b}_1+169\bar{b}_2+2874\bar{b}_3+6270\bar{b}_4+\frac{205\pi^2}{6}-\frac{549221}{360}\right)\right],\\
c_F &= \frac{463}{3274425\pi^3 f_\pi^4 M_\pi^7}\left[1+\frac{5M_\pi^2}{29632\pi^2 f_\pi^2}\left(675\bar{b}_1+\frac{7079}{5}\bar{b}_2+14341\bar{b}_3+\frac{134517}{5}\bar{b}_4+200\pi^2+\frac{2301641}{900}\right)\right].
\end{aligned}
\tag{B1}
$$

$$c_{S0} = \frac{1}{3456\pi^3 f_\pi^4 M_\pi}\Bigg[792\bar{b}_3 + 1224\bar{b}_4 - 253 + \frac{M_\pi^2}{\pi^2 f_\pi^2}\left(-\frac{219}{8}\bar{b}_1 - 59\bar{b}_2 + \frac{2893}{5}\bar{b}_3 + \frac{4781}{5}\bar{b}_4 + 486\bar{b}_5 - 90\bar{b}_6 - \frac{91589\pi^2}{384} + \frac{685061}{480}\right)\Bigg],$$

$$c_{S2} = \frac{1}{8640\pi^3 f_\pi^4 M_\pi}\left[360\bar{b}_3 + 2520\bar{b}_4 - 19 + \frac{M_\pi^2}{\pi^2 f_\pi^2}\left(\frac{267}{8}\bar{b}_1 + 31\bar{b}_2 - 106\bar{b}_3 - 956\bar{b}_4 - 360\bar{b}_6 + \frac{1049\pi^2}{12} - \frac{75997}{96}\right)\right],$$

$$c_P = \frac{-23}{6720\pi^3 f_\pi^4 M_\pi^3}\left[1 - \frac{M_\pi^2}{3726\pi^2 f_\pi^2}\left(\frac{729}{2}\bar{b}_1 - \frac{405}{4}\bar{b}_2 - \frac{23823}{2}\bar{b}_3 - \frac{36261}{2}\bar{b}_4 + 5103(\bar{b}_5 + \bar{b}_6) + \frac{1551\pi^2}{40} + 21037\right)\right], \tag{B2}$$

$$c_{D0} = \frac{499}{264600\pi^3 f_\pi^4 M_\pi^5}\left[1 + \frac{M_\pi^2}{17964\pi^2 f_\pi^2}\left(135\bar{b}_1 + \frac{4761}{2}\bar{b}_2 + 19701\bar{b}_3 - 29079\bar{b}_4 + \frac{2667\pi^2}{4} - \frac{18331}{10}\right)\right],$$

$$c_{D2} = \frac{127}{105840\pi^3 f_\pi^4 M_\pi^5}\left[1 + \frac{M_\pi^2}{2540\pi^2 f_\pi^2}\left(102\bar{b}_1 + \frac{839}{2}\bar{b}_2 + 5053\bar{b}_3 + 5421\bar{b}_4 - \frac{73\pi^2}{12} - \frac{342941}{360}\right)\right],$$

[1] S. Weinberg, Physica **A96** (1979) 327.
[2] J. Gasser and H. Leutwyler, Annals Phys. **158** (1984) 142;
[3] M. Knecht, B. Moussallam, J. Stern, N. H. Fuchs, Nucl. Phys. **B457**, 513-576 (1995). [hep-ph/9507319].
[4] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, M. E. Sainio, Phys. Lett. **B374**, 210-216 (1996). J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, M. E. Sainio, Nucl. Phys. **B508**, 263-310 (1997) [Erratum-ibid. B **517**, 639 (1998)] [hep-ph/9707291].
[5] G. Colangelo, S. Durr, A. Juttner, L. Lellouch, H. Leutwyler, V. Lubicz, S. Necco and C. T. Sachrajda *et al.*, Eur. Phys. J. C **71**, 1695 (2011) [arXiv:1011.4408 [hep-lat]].
[6] D. Espriu, E. de Rafael, J. Taron, Nucl. Phys. **B345**, 22-56 (1990). F. J. Llanes-Estrada, P. De A. Bicudo, Phys. Rev. **D68**, 094014 (2003).
[7] G. Ecker, J. Gasser, A. Pich, E. de Rafael, Nucl. Phys. **B321**, 311 (1989). J. F. Donoghue, C. Ramirez, G. Valencia, Phys. Rev. **D39** (1989) 1947.
[8] J. Gasser and H. Leutwyler, Nucl. Phys. B **250** (1985) 465.
[9] G. Amoros, J. Bijnens and P. Talavera, Nucl. Phys. B **585**, 293 (2000) [Erratum-ibid. B **598**, 665 (2001)] [hep-ph/0003258].
[10] C. Riggenbach, J. Gasser, J. F. Donoghue and B. R. Holstein, Phys. Rev. D **43**, 127 (1991).
[11] G. Colangelo, J. Gasser and H. Leutwyler, Nucl. Phys. B **603**, 125 (2001) [arXiv:hep-ph/0103088].
[12] S. R. Beane *et al.* [NPLQCD Collaboration], Phys. Rev. D **85**, 034505 (2012) [arXiv:1107.5023 [hep-lat]].
[13] R. Garcia-Martin, R. Kaminski, J. R. Pelaez, J. Ruiz de Elvira, F. J. Yndurain, Phys. Rev. **D83**, 074004 (2011).
[14] J. R. Batley *et al.* [NA48-2 Collaboration], Eur. Phys. J. **C70**, 635-657 (2010).
[15] R. Kaminski, J. R. Pelaez and F. J. Yndurain, Phys. Rev. D **77**, 054015 (2008)
[16] J. R. Pelaez and F. J. Yndurain, Phys. Rev. D **71**, 074016 (2005)